

Spin-orbital composition in relativistic many-fermion systems

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The interplay of spins and orbital angular moments of the fermions play an important role for the structure of the many-fermion systems like atoms, nuclei, nucleons (baryons) or mesons. We start our study from the one-fermion eigenstates of angular momentum represented by the spinor spherical harmonics. Afterwards we study the properties of many-fermion states resulting from a multiple angular momentum composition of the one-fermion states, giving the total angular momentum $J = \langle L \rangle + \langle S \rangle$, which is identified with the spin of the composite particle. We demonstrate how the composition rules affect the relativistic interplay between the sums of the spins $\langle S \rangle$ and orbital angular moments $\langle L \rangle$ of the constituents, which collectively generate the spin of composite particle. It is suggested that in a relativistic case, when the masses of the constituent fermions are much less than their energy (in the rest frame of the composite particle), then the spin of the composite particle is dominated by the orbital angular moments $\langle L \rangle$ of the constituents, while $|\langle S \rangle| \leq J/3$. A special attention is paid to the case $J = 1/2$ that is related to the spin of proton generated by the composition of spins and orbital angular moments of the quarks.

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1. EIGENSTATES OF ANGULAR MOMENTUM

The solutions of free Dirac equation represented by eigenstates of the total angular momentum (AM) with quantum numbers j, j_z are the spinor spherical harmonics [1–3], which in the *momentum representation* reads

$$|j, j_z\rangle = \Phi_{jl_p j_z}(\omega) = \frac{1}{\sqrt{2\epsilon}} \begin{pmatrix} \sqrt{\epsilon + m} \Omega_{jl_p j_z}(\omega) \\ -\sqrt{\epsilon - m} \Omega_{j\lambda_p j_z}(\omega) \end{pmatrix}, \quad (1)$$

where ω represents the polar and azimuthal angles (θ, φ) of the momentum \mathbf{p} with respect to the quantization axis z , $l_p = j \pm 1/2$, $\lambda_p = 2j - l_p$ (l_p defines the parity), energy $\epsilon = \sqrt{\mathbf{p}^2 + m^2}$, and

$$\Omega_{jl_p j_z}(\omega) = \begin{pmatrix} \sqrt{\frac{j+j_z}{2j}} Y_{l_p, j_z-1/2}(\omega) \\ \sqrt{\frac{j-j_z}{2j}} Y_{l_p, j_z+1/2}(\omega) \end{pmatrix}; \quad l_p = j - \frac{1}{2}, \quad (2)$$

$$\Omega_{j\lambda_p j_z}(\omega) = \begin{pmatrix} -\sqrt{\frac{j-j_z+1}{2j+2}} Y_{l_p, j_z-1/2}(\omega) \\ \sqrt{\frac{j+j_z+1}{2j+2}} Y_{l_p, j_z+1/2}(\omega) \end{pmatrix}; \quad l_p = j + \frac{1}{2}.$$

In a relativistic case the quantum numbers of spin and orbital angular momentum (OAM) are not conserved separately, but only the total AM j and its projection $j_z = s_z + l_z$ can be conserved. The complete wave function reads

$$\Psi_{jl_p j_z}(\epsilon, \omega) = \phi_j(\epsilon) \Phi_{jl_p j_z}(\omega). \quad (3)$$

The function $\phi_j(\epsilon)$ or its equivalent representation (7) is the amplitude of probability that the fermion has energy ϵ . In fact the main results in this note depend only on the probability distribution $a_j^*(\epsilon) a_j(\epsilon)$ via the parameters (16). The spinors (1) are normalized as

$$\int \Phi_{j'l'_p j'_z}^+(\omega) \Phi_{jl_p j_z}(\omega) d\omega = \delta_{j'j} \delta_{l'_p l_p} \delta_{j'_z j_z}, \quad (4)$$

j, j_z	$P_{j, j_z}(\omega)$
$\frac{1}{2}, \frac{1}{2}$	1
$\frac{3}{2}, \frac{3}{2}$	$\frac{3-3\cos 2\theta}{4}$
$\frac{3}{2}, \frac{1}{2}$	$\frac{5+3\cos 2\theta}{4}$
$\frac{5}{2}, \frac{5}{2}$	$\frac{45-60\cos 2\theta+15\cos 4\theta}{64}$

TABLE I: The examples of the distributions (8). The common factor $1/4\pi$ is omitted.

where $d\omega = d\cos\theta d\varphi$. Then the normalization

$$\int \Psi_{j'l'_p j'_z}^+(\epsilon, \omega) \Psi_{jl_p j_z}(\epsilon, \omega) d^3\mathbf{p} = \delta_{j'j} \delta_{l'_p l_p} \delta_{j'_z j_z} \quad (5)$$

implies the condition for the amplitude ϕ_j ,

$$\int \phi_j^*(\epsilon) \phi_j(\epsilon) p^2 dp = 1. \quad (6)$$

In the next discussion it will be convenient also to use the alternative representation, which differs in normalization,

$$a_j(\epsilon) = \frac{\phi_j(\epsilon)}{2\sqrt{\pi}}; \quad \int a_j^*(\epsilon) a_j(\epsilon) d^3\mathbf{p} = 1. \quad (7)$$

1.1. Angular moments of one-fermion states

A few examples of the corresponding probability distribution

$$P_{j, j_z}(\omega) = \Phi_{jl_p j_z}^+(\omega) \Phi_{jl_p j_z}(\omega); \quad \int P_{j, j_z}(\omega) d\omega = 1, \quad (8)$$

are given in Table I. These distributions does not depend on the parameters φ and $l_p = j \pm 1/2$. The lowest value $j = 1/2$ generates rotational symmetry of the probability distribution, but for higher $j = 3/2, 5/2, \dots$

the distribution has axial symmetry only. The states (1) are not eigenstates of spin and OAM; nevertheless, one can always calculate the mean values of corresponding operators

$$s_z = \frac{1}{2} \begin{pmatrix} \sigma_z & 0 \\ 0 & \sigma_z \end{pmatrix}, \quad l_z = -i \left(p_x \frac{\partial}{\partial p_y} - p_y \frac{\partial}{\partial p_x} \right). \quad (9)$$

The related matrix elements are given by the relations [4]:

$$\langle s_z \rangle_{j,j_z} = \int \Phi_{j l_p j_z}^+ s_z \Phi_{j l_p j_z} d\omega = \frac{1 + (2j+1)\mu}{4j(j+1)} j_z, \quad (10)$$

$$\begin{aligned} \langle l_z \rangle_{j,j_z} &= \int \Phi_{j l_p j_z}^+ l_z \Phi_{j l_p j_z} d\omega \\ &= \left(1 - \frac{1 + (2j+1)\mu}{4j(j+1)} \right) j_z, \end{aligned}$$

in which we have denoted

$$\mu = \pm \frac{m}{\epsilon}, \quad (11)$$

where the sign (\pm) corresponds to $l_p = j \mp 1/2$. The relations imply that in the nonrelativistic limit, when $\mu \simeq \pm 1$, we get for signs (\pm) correspondingly,

$$\langle s_z \rangle_{j,j_z} = \left\{ \frac{j_z}{2j} \right\}, \quad \langle l_z \rangle_{j,j_z} = \left\{ \begin{pmatrix} 1 - \frac{1}{2j} \end{pmatrix} j_z \right\} \quad (12)$$

and in the relativistic case, when $\mu \rightarrow 0$, we have

$$\langle s_z \rangle_{j,j_z} = \frac{j_z}{4j(j+1)}, \quad \langle l_z \rangle_{j,j_z} = \left(1 - \frac{1}{4j(j+1)} \right) j_z. \quad (13)$$

The last two relations imply

$$\begin{aligned} \left| \langle s_z \rangle_{j,j_z} \right| &\leq \frac{1}{4(j+1)} \leq \frac{1}{6}, \\ \left| \langle l_z \rangle_{j,j_z} \right| &\leq \frac{1}{4j^2 + 4j - 1} \leq \frac{1}{2}. \end{aligned} \quad (14)$$

For the complete wave function (3), the relations (10) are modified as

$$\langle \langle s_z \rangle \rangle_{j,j_z} = \int \Psi_{j l_p j_z}^+ s_z \Psi_{j l_p j_z} d^3 \mathbf{p} = \frac{1 + (2j+1)\langle \mu_j \rangle}{4j(j+1)} j_z, \quad (15)$$

$$\begin{aligned} \langle \langle l_z \rangle \rangle_{j,j_z} &= \int \Psi_{j l_p j_z}^+ l_z \Psi_{j l_p j_z} d^3 \mathbf{p} \\ &= \left(1 - \frac{1 + (2j+1)\langle \mu_j \rangle}{4j(j+1)} \right) j_z, \end{aligned}$$

where

$$\langle \mu_j \rangle = \pm \int a_j^*(\epsilon) a_j(\epsilon) \frac{m}{\epsilon} d^3 \mathbf{p}, \quad |\langle \mu_j \rangle| \leq 1. \quad (16)$$

1.2. Many-fermion states

The system of fermions (or arbitrary particles) generating the state with quantum numbers J, J_z can be represented by the combination of one-particle states. For example the pair of states j_1, j_2 can generate the states

$$|(j_1, j_2) J, J_z\rangle \quad (17)$$

$$\begin{aligned} &= \sum_{j_{z1}=-j_1}^{j_1} \sum_{j_{z2}=-j_2}^{j_2} \langle j_1, j_{z1}, j_2, j_{z2} | J, J_z \rangle |j_1, j_{z1}\rangle |j_2, j_{z2}\rangle; \\ &j_{z1} + j_{z2} = J_z, \quad |j_1 - j_2| \leq J \leq j_1 + j_2, \end{aligned} \quad (18)$$

where $\langle j_1, j_{z1}, j_2, j_{z2} | J, J_z \rangle$ are Clebsch-Gordan coefficients, which are nonzero if the conditions (18) are satisfied. In this way one can repeat the composition and obtain the many-particle eigenstates of resulting J, J_z

$$\begin{aligned} &|(j_1, j_2, \dots, j_n)_c J, J_z\rangle \\ &= \sum_{j_{z1}=-j_1}^{j_1} \sum_{j_{z2}=-j_2}^{j_2} \dots \sum_{j_{zn}=-j_n}^{j_n} c_j |j_1, j_{z1}\rangle |j_2, j_{z2}\rangle \dots |j_n, j_{zn}\rangle, \end{aligned} \quad (19)$$

where the coefficients c_j are a product of the Clebsch-Gordan coefficients

$$\begin{aligned} c_j &= \langle j_1, j_{z1}, j_2, j_{z2} | J_3, J_{z3} \rangle \langle J_3, J_{z3}, j_3, j_{z3} | J_4, J_{z4} \rangle \\ &\dots \langle J_n, J_{zn}, j_n, j_{zn} | J, J_z \rangle. \end{aligned} \quad (20)$$

Let us remark that the set j_1, j_2, \dots, j_n does not define the resulting state unambiguously. The result depends on the pattern of their composition, e.g.

$$\begin{aligned} &(((j_1 \oplus j_2)_{J_1} \oplus j_3)_{J_2} \oplus j_4)_{J_3}, \\ &(((j_1 \oplus j_2)_{J_1} \oplus (j_3 \oplus j_4)_{J_2})_{J_3} \oplus j_5)_{J_4}, \end{aligned} \quad (21)$$

where J_k represent intermediate AMs corresponding to the steps of composition:

$$j_1 \oplus j_2 = J_1, \quad J_1 \oplus j_3 = J_2, \quad J_2 \oplus j_4 = J. \quad (22)$$

Each binary composition " \oplus " is defined by Eq. (17). Different composition patterns are in (19) symbolically expressed by the subscript c . Apparently, the number of patterns increases with n very rapidly; however, in a real scenario with an interaction one can expect their probabilities will differ. The case $n = 3$ will be illustrated in more detail below.

From now we discuss only the composed states with resulting $J = J_z = 1/2$ ($J_z = -1/2$ gives the equivalent results). The corresponding n -fermion state (n is odd)

$$\Phi_{c,1/2,1/2}(\omega_1, \omega_2, \dots, \omega_n) = |(j_1, j_2, \dots, j_n)_c 1/2, 1/2\rangle, \quad (23)$$

or alternatively

$$\begin{aligned} \Psi_{c,1/2,1/2} &= \phi_{j_1}(\epsilon_1) \phi_{j_2}(\epsilon_2) \dots \phi_{j_n}(\epsilon_n) \\ &\times \Phi_{c,1/2,1/2}(\omega_1, \omega_2, \dots, \omega_n) \end{aligned} \quad (24)$$

generate the n -dimensional angular distribution

$$P_c(\omega_1, \omega_2, \dots, \omega_n) = \Phi_{c,1/2,1/2}^+ \Phi_{c,1/2,1/2}, \quad (25)$$

from which the corresponding average one-fermion distributions are obtained as

$$p_{c,k}(\omega_k) = \int P_c(\omega_1, \omega_2, \dots, \omega_n) \prod_{i \neq k} d\omega_i, \quad (26)$$

which gives [4]:

$$p_{c,k}(\omega) = \frac{1}{4\pi}. \quad (27)$$

It follows that the distribution

$$P_c(\omega) = \sum_{k=1}^n p_{c,k}(\omega) = \frac{n}{4\pi}, \quad (28)$$

which is generated by the state (23) has rotational symmetry similar to the distribution $P_{1/2,1/2}$ generated by the one-fermion state in Table I. Therefore the angular probability distribution $P_c(\omega)$ related to the state $J = 1/2$ has rotational symmetry regardless of the number of involved particles. This rule suggests that e.g. in a nucleus $J = 1/2$, the probability distribution of nucleons, separately for protons and neutrons, has in the momentum space rotational symmetry. Spherical symmetry of probability distribution in the momentum space apparently implies spherical symmetry in coordinate representation.

What can be said about the mean values of the spin and OAM contributions

$$\begin{aligned} \langle \mathbb{S}_z \rangle_{c,1/2,1/2} &= \langle s_{z1} + s_{z2} + \dots + s_{zn} \rangle_c, \\ \langle \mathbb{L}_z \rangle_{c,1/2,1/2} &= \langle l_{z1} + l_{z2} + \dots + l_{zn} \rangle_c, \\ \langle \mathbb{S}_z \rangle_{c,1/2,1/2} + \langle \mathbb{L}_z \rangle_{c,1/2,1/2} &= \frac{1}{2}, \end{aligned} \quad (29)$$

corresponding to the state (23)? Now we will discuss this question in more detail for the case $n = 3$.

1.2.1. Three-fermion states

There are three patterns for composition of the three AMs j_a, j_b, j_c :

$$((j_a \oplus j_b)_{J_c} \oplus j_c)_{1/2}; \quad abc = 123, 312, 231. \quad (30)$$

Corresponding states are

$$\begin{aligned} &\Phi_{c,1/2,1/2}(\omega_1, \omega_2, \omega_3) \\ &= \sum_{j_{z1}=-j_1}^{j_1} \sum_{j_{z2}=-j_2}^{j_2} \sum_{j_{z3}=-j_3}^{j_3} \langle j_a, j_{za}, j_b, j_{zb} | J_c, J_{zc} \rangle \\ &\times \langle J_c, J_{zc}, j_c, j_{zc} | 1/2, 1/2 \rangle | j_1, j_{z1} \rangle | j_2, j_{z2} \rangle | j_3, j_{z3} \rangle. \end{aligned} \quad (31)$$

j_1	j_2	j_3	$\langle S_z \rangle_3$	$\langle S_z \rangle_2$	$\langle S_z \rangle_1$	$\langle S_z \rangle_3$	$\langle S_z \rangle_2$	$\langle S_z \rangle_1$
$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1+2\tilde{\mu}}{6}$	$\frac{1+2\tilde{\mu}}{6}$	$\frac{1+2\tilde{\mu}}{6}$	$\frac{1+2\tilde{\mu}}{6}$	$\frac{1+2\tilde{\mu}}{6}$	$\frac{1+2\tilde{\mu}}{6}$
$\frac{3}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	\times	\times	$\frac{-1}{18}$	$\frac{-1}{18}$	$\frac{-1}{18}$	\times
$\frac{3}{2}$	$\frac{3}{2}$	$\frac{1}{2}$	$\frac{1+2\tilde{\mu}}{6}$	$\frac{1+3\tilde{\mu}}{18}$	$\frac{1+3\tilde{\mu}}{18}$	$\frac{-1+6\tilde{\mu}}{90}$	$\frac{3+7\tilde{\mu}}{30}$	$\frac{3+7\tilde{\mu}}{30}$
$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{1+4\tilde{\mu}}{30}$	$\frac{1+4\tilde{\mu}}{30}$	$\frac{1+4\tilde{\mu}}{30}$	$\frac{1+4\tilde{\mu}}{30}$	$\frac{1+4\tilde{\mu}}{30}$	$\frac{1+4\tilde{\mu}}{30}$
$\frac{5}{2}$	$\frac{3}{2}$	$\frac{1}{2}$	\times	\times	$\frac{-5-4\tilde{\mu}}{90}$	$\frac{-5-4\tilde{\mu}}{90}$	$\frac{-5-4\tilde{\mu}}{90}$	\times
$\frac{5}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{5+17\tilde{\mu}}{90}$	$\frac{5+17\tilde{\mu}}{90}$	$\frac{-1+2\tilde{\mu}}{90}$	$\frac{-1+29\tilde{\mu}}{630}$	$\frac{-1+29\tilde{\mu}}{630}$	$\frac{41+134\tilde{\mu}}{630}$
$\frac{5}{2}$	$\frac{5}{2}$	$\frac{1}{2}$	$\frac{1+2\tilde{\mu}}{6}$	$\frac{13+38\tilde{\mu}}{270}$	$\frac{13+38\tilde{\mu}}{270}$	$\frac{-23+2\tilde{\mu}}{630}$	$\frac{31+74\tilde{\mu}}{378}$	$\frac{31+74\tilde{\mu}}{378}$
$\frac{5}{2}$	$\frac{5}{2}$	$\frac{3}{2}$	$\frac{29+104\tilde{\mu}}{630}$	$\frac{23+152\tilde{\mu}}{1890}$	$\frac{23+152\tilde{\mu}}{1890}$	$\frac{-1+8\tilde{\mu}}{210}$	$\frac{55+232\tilde{\mu}}{1890}$	$\frac{55+232\tilde{\mu}}{1890}$
$\frac{5}{2}$	$\frac{5}{2}$	$\frac{5}{2}$	$\frac{1+6\tilde{\mu}}{70}$	$\frac{1+6\tilde{\mu}}{70}$	$\frac{1+6\tilde{\mu}}{70}$	$\frac{1+6\tilde{\mu}}{70}$	$\frac{1+6\tilde{\mu}}{70}$	$\frac{1+6\tilde{\mu}}{70}$

TABLE II: Mean values $\langle \mathbb{S}_z \rangle_c$ of three-fermion states $| (j_1, j_2, j_3, J_c) 1/2, 1/2 \rangle$ with $J_c = j_c - 1/2$ and $J_c = j_c + 1/2$ (columns 4,5,6 and 7,8,9; $c = 3, 2, 1$) [see the first relation (32) and (34)]. The symbol \times denotes configuration for which the second condition (32) is not satisfied.

The conditions (18) give at most two possibilities for the intermediate values J_c , which must satisfy

$$J_c = j_c \pm 1/2, \quad |j_a - j_b| \leq J_c \leq j_a + j_b. \quad (32)$$

At the same time it holds

$$j_{z1} + j_{z2} + j_{z3} = 1/2, \quad j_{za} + j_{zb} = J_{zc}. \quad (33)$$

In this way two possible values J_c in three patterns (30) give six possibilities to create the state (31). Further, if we take into account two possible values $l_p = j \pm 1/2$ for each one-fermion state in (31) and defined by (1), then in general the total number of generated three-fermion states is $6 \times 2^3 = 48$. Due to orthogonality of the terms in sum (31) the three-fermion mean values (29) are calculated as

$$\begin{aligned} &\langle \mathbb{S}_z \rangle_{c,1/2,1/2} \\ &= \sum_{j_{z1}=-j_1}^{j_1} \sum_{j_{z2}=-j_2}^{j_2} \sum_{j_{z3}=-j_3}^{j_3} \langle j_a, j_{za}, j_b, j_{zb} | J_c, J_{zc} \rangle^2 \\ &\times \langle J_c, J_{zc}, j_c, j_{zc} | 1/2, 1/2 \rangle^2 (\langle s_{za} \rangle + \langle s_{zb} \rangle + \langle s_{zc} \rangle) \end{aligned} \quad (34)$$

and similarly for $\langle \mathbb{L}_z \rangle_{c,1/2,1/2}$. Corresponding one-fermion values $\langle s_{z..} \rangle$ and $\langle l_{z..} \rangle$ are given by the relations (10). The results for a set of input values j_1, j_2, j_3 and $l_{pk} = j_k - 1/2$ are listed in Table II and the results corresponding to remaining sets $l_{pk} = j_k \pm 1/2$ are similar and differ only in terms proportional to $\tilde{\mu}$. Since

$$\begin{aligned} \langle \mathbb{S}_z \rangle_{c,1/2,1/2} &= -\langle \mathbb{S}_z \rangle_{c,1/2,-1/2}, \\ \langle \mathbb{S}_z \rangle_{c,1/2,\pm 1/2} + \langle \mathbb{L}_z \rangle_{c,1/2,\pm 1/2} &= \pm 1/2, \end{aligned} \quad (35)$$

we present only $\langle \mathbb{S}_z \rangle_c \equiv \langle \mathbb{S}_z \rangle_{c,1/2,1/2}$. The meaning of the parameter $\tilde{\mu}$ is as follows:

(1) If one assumes the same parameter μ (11) for the three fermions in the state (31), then $\tilde{\mu} = \mu$.

(2) In a general case, the complete wave function

$$\Psi_{c,1/2,1/2} = \phi_{j_1}(\epsilon_1) \phi_{j_2}(\epsilon_2) \phi_{j_3}(\epsilon_3) \times \Phi_{c,1/2,1/2}(\omega_1, \omega_2, \omega_3) \quad (36)$$

gives instead of (16) a more complicated expression [4]

$$\tilde{\mu} = f_c(\langle\mu_1\rangle, \langle\mu_2\rangle, \langle\mu_3\rangle, j_1, j_2, j_3), \quad (37)$$

where the parameters $\langle\mu_i\rangle$ are defined by Eq. (16). The expression is simplified for $\langle\mu_1\rangle = \langle\mu_2\rangle = \langle\mu_3\rangle = \langle\mu\rangle$,

$$f_c(\langle\mu\rangle, \langle\mu\rangle, \langle\mu\rangle, j_1, j_2, j_3) = \langle\mu\rangle. \quad (38)$$

Obviously the many-fermion system with $J = J_z = 1/2$ can be treated as a composed particle of the spin $1/2$. This spin is generated by the spins and OAMs of the involved fermions. The relative weights of the spin and OAM contributions vary depending not only on the intrinsic values j_1, j_2, j_3 and the pattern of composition, but also on the mass-motion parameter $\tilde{\mu}$. The data in the table suggest that for any configuration in the relativistic limit $\tilde{\mu} \rightarrow 0$, we have

$$|\langle S_z \rangle| \leq \frac{1}{6} \quad (39)$$

similar to the case of the one-fermion states (14).

The table illustrates a complexity of the AM composition even for only three fermions. Is there a simple rule like (39) for $n > 3$? First, let us consider the composition

$$\Psi_{c,1/2,1/2} = |(j_1, j_2, \dots, j_n)_c 1/2, 1/2\rangle, \quad (40)$$

where all one-fermion AMs are the same, $j_i = j$ (like the rows 1,4,9 in the table). The corresponding spin reads

$$\langle S_z \rangle = \frac{1 + (2j + 1)\tilde{\mu}}{8j(j + 1)} \quad (41)$$

regardless of n and details of composition. The proof of this relation is given in [4]. Apparently for $\tilde{\mu} \rightarrow 0$, the relation (39) is again satisfied. The situation with the composition of different AMs is getting much more complex for increasing n . However, an average value of the spin over all possible composition patterns of the state $|(j_1, j_2, \dots, j_n)_c 1/2, 1/2\rangle$ appears to safely satisfy (39). This is the case when there is no (e.g., dynamical) preference among various composition patterns.

Let us illustrate a possible role of the composition patterns by the simple example $j_1, j_2, j_3 = 1/2$. Equation (31) gives the three states corresponding to $J_c = 1$,

$$\Psi_{abc,1/2,1/2} = \frac{\phi_{abc}}{\sqrt{6}} (|-1/2, 1/2, 1/2\rangle + |1/2, -1/2, 1/2\rangle - 2|1/2, 1/2, -1/2\rangle), \quad (42)$$

where

$$\phi_{abc} = \phi_a(\epsilon_a) \phi_b(\epsilon_b) \phi_c(\epsilon_c). \quad (43)$$

The indices abc define the composition in accordance with (30), and AM states are defined correspondingly, $|j_{za}, j_{zb}, j_{zc}\rangle$. The other three states correspond to $J_c = 0$,

$$\Psi_{abc,1/2,1/2} = \frac{\phi_{abc}}{\sqrt{2}} (|1/2, -1/2, 1/2\rangle - |-1/2, 1/2, 1/2\rangle). \quad (44)$$

The nonrelativistic proton SU(6) wave function in a standard notation reads:

$$|p \uparrow\rangle = \frac{1}{\sqrt{2}} \left\{ \frac{1}{\sqrt{6}} |duu + udu - 2uud\rangle \times \frac{1}{\sqrt{6}} |\downarrow\uparrow\uparrow + \uparrow\downarrow\uparrow - 2\uparrow\uparrow\downarrow\rangle + \frac{1}{\sqrt{2}} |duu - udu\rangle \frac{1}{\sqrt{2}} |\downarrow\uparrow\uparrow - \uparrow\downarrow\uparrow\rangle \right\}. \quad (45)$$

The comparison (42)–(44) with (45) suggests that the SU(6) wave function after substitution

$$\phi_a(\epsilon_a) = u_1, \quad \phi_b(\epsilon_b) = u_2, \quad \phi_c(\epsilon_c) = d$$

can be obtained as the superposition of wave functions generated by the AM compositions

$$((u_1 \oplus u_2)_J \oplus d)_{1/2}, \quad ((d \oplus u_1)_J \oplus u_2)_{1/2}, \quad ((u_2 \oplus d)_J \oplus u_1)_{1/2} \quad (46)$$

for $J = 1, 2$.

2. CONCLUSION

Our study was focused on the many-fermion system carrying spin $J = 1/2$, however the relation (39) can be generalized for arbitrary spin J

$$|\langle S_z \rangle| \leq \frac{J}{3} \quad (47)$$

provided that:

- (1) the intrinsic motion of the fermions inside the system (composite particle) is relativistic ($\tilde{\mu} \rightarrow 0$),
- (2) mean value $\langle S_z \rangle$ include an averaging over possible composition patterns (if the number of fermions $n \geq 3$)

The ratio $\tilde{\mu} = \langle m/\epsilon \rangle$ is of key importance, since it controls a "contraction" of the spin component (47), which is compensated by the OAM. It is a pure effect of relativistic quantum mechanics. The obtained results for $J = 1/2$ have been applied to the description of the proton spin structure in terms of the structure functions g_1 and g_2 in Ref. [4], where we have suggested the proton studied at polarized deep inelastic scattering is an ideal instrument for the study of this relativistic effect.

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